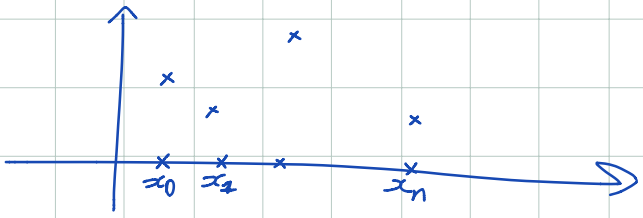
6.1 Polynomial Interpolation

Goal: Given a set of $n+1$ data points (x_i, y_i)



we seek a polynomial p with lowest possible degree so that $p(x_i) = y_i$

We say that p interpolates the data.

Theorem: If $x_0, x_1, \dots, x_n$ are distinct real numbers then for arbitrary $y_0, \dots, y_n$ $\exists$ a unique polynomial P_n of degree n or less s.t.

$$P_n(x_i) = y_i \text{ for } i = 0, 1, \dots n$$

Proof: (1) Existence:

We'll do it by induction

- for $n=0$ we can always find P_0 (const. fct.) s.t. $P_0(x_0) = y_0$
- Suppose P_{k-1} satisfies $P_{k-1}(x_i) = y_i$ for $i = 0, 1, \dots, k-1$

Let $P_k(x) = P_{k-1}(x) + C(x-x_0)(x-x_1)\cdots(x-x_{k-1})$

($(x-x_0)\cdots(x-x_{k-1})$: degree $\le k$)

To find C we just solve $P_k(x_k) = y_k$

so $P_k(x_k) = P_{k-1}(x_k) + C(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})$ $\Rightarrow$ can solve!

($P_k(x_k)$: y_k; $P_{k-1}(x_k)$: known bec P_{k-1} is known; $(x_k-x_0)\cdots(x_k-x_{k-1})$: all $\neq 0$ bec x_i's distinct)

we also note that $P_k(x_i) = P_{k-1}(x_i) = y_i$ for $i = 0, \dots, k-1$

so P_k interpolates the data (x_i, y_i), $i = 0, \dots, k$

Uniqueness: Suppose $\exists q_n$ that interpolates the data

then $q_n(x_i) - P_n(x_i) = 0$ for $i = 0, \dots, n$

$\Rightarrow (q_n - P_n)(x_i) = 0$ → $n+1$ of these

poly of degree $\leq n$ with $n+1$ zeros $\Rightarrow q_n - P_n = 0$ ▨

Newton form of the interpolating polynomial

From the proof: $P_k(x) = P_{k-1}(x) + c_k(x - x_0)\dots(x - x_{k-1})$

$= \dots$

$= c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \dots$

$+ c_k(x - x_0)\dots(x - x_{k-1})$

In short form: $\boxed{P_k(x) = \sum_{i=0}^{k-1} c_i \prod_{j=0}^{i-1} (x - x_j)}$

↑ interp. polynomials in Newton form.

Lagrange form of interpolating polynomial

We can write our polynomial as

$$P_n(x) = \sum_{k=0}^{n} y_k \, \ell_k(x)$$

→ polynomials degree $\leq n$ depending on the nodes $x_0 — x_n$ not the ordinates $y_0 — y_n$

Specifically we may choose $\ell_k(x_i) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$

$$= \delta_{ik}$$

this can be done, e.g., by setting

$$\ell_0(x) = c \prod_{i=1}^{n} (x - x_i) \Rightarrow c = \frac{\overbrace{\ell_0(x_0)}^{1}}{\prod_{i=1}^{n} (x_0 - x_i)} = \prod_{i=1}^{n} (x_0 - x_i)^{-1}$$

$$\Rightarrow \quad \ell_0(x) = \prod_{i=1}^{n} \frac{x - x_i}{x_0 - x_i}$$

Similarly

$$\ell_j(x) = \prod_{\substack{i \neq j \\ i=0}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$

cardinal functions

$$P_n(x) = \sum_{i=0}^{n} y_i \, \ell_i(x)$$

Lagrange form

Example Given

x	5	-7	-6	0
y	1	-23	-54	954

Find the cardinal functions and the Lagrange form of the interpolating polynomial

$$l_0(x) = \frac{x-(-7)}{5-(-7)} \cdot \frac{x-(-6)}{5-(-6)} \cdot \frac{x-0}{5-0} = \frac{(x+6)(x+7)x}{660}$$

$$l_1(x) = \frac{x-5}{-7-5} \cdot \frac{x-(-6)}{-7-(-6)} \cdot \frac{x-0}{-7-0} = \frac{(x-5)(x+6)x}{-84}$$

$l_2(x) = \dots$

$l_3(x) = \dots$

$\Rightarrow P_3(x) = l_0(x) - 23 l_1(x) - 54 l_2(x) - 954 l_3(x)$

Theorem (error in polynomial interp)

Let $f \in C^{n+1}[a,b]$ & P be a poly. of degree $\le n$ that interpolates f at $n+1$ distinct pts $x_0, x_1, \dots x_n \in [a,b]$.

Then $\forall x \in [a,b]$, $\exists \xi_x \in (a,b)$ s.t.

$$f(x) - P(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

(1)

Exercise: read the proofs.

HW: 6.1 : 1(a,b), 9, 14, 22

6.2 Divided Differences

Recall: $$P_k(x) = \sum_{i=0}^{k-1} c_i \prod_{j=0}^{i-1} (x - x_j)$$

↑ interp. polynomials in Newton form.

If we call $q_0(x) = 1$

$q_1(x) = x - x_0$

$q_2(x) = (x - x_1)(x - x_0)$

$\vdots$

$q_n(x) = \prod_{i=0}^{n-1} (x - x_i)$

then $P_n(x) = \sum_{j=0}^{n} c_j q_j(x)$

Recall: the c_j's are what we need to find.

We have $n+1$ linear equations:

$$\sum_{j=0}^{n} c_j \underbrace{q_j(x_i)}_{a_{ij}} = \underbrace{f(x_i)}_{y_i}, \quad i = 0, 1, \ldots, n+1$$

In matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_1-x_0 & 0 & \cdots & 0 \\ 1 & x_1-x_0 & (x_2-x_0)(x_2-x_1) & \cdots & 0 \\ \vdots & & & \ddots & \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

So: c_0 depends on y_0, we write $c_0 = f[x_0]$

c_1 depends on y_0, y_1, we write $c_1 = f[x_0, x_1]$

$\vdots$

c_n depends on $y_0, y_1, \dots y_n$, we write $c_n = f[x_0, x_1, \dots, x_n]$

old notation

Let's compute a couple of these:

- $f[x_0] = f(x_0) \quad (= c_0)$
- $f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (= c_1)$

& $P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$

Theorem: $f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$

Proof: exercise/read it

So • $f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

This makes it easy to compute the $f[\ ]$'s using a divided differences table

x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	$f[x_1]$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
x_2	$f[x_2]$	$f[x_2, x_3]$		
x_3	$f[x_3]$			

e.g. Use divided differences to find the Newton interpolating polynomial for

x	3	1	5	6
$f(x)$	1	−3	2	4

Soln: Step 1 (table)

x	$f[x]$			
3	1	$\frac{-3-1}{1-3} = 2$	$\frac{5/4 - 2}{5-3} = -3/8$	$\frac{3/20 - (-3/8)}{6-3} = \frac{7}{40}$
1	−3	$\frac{2-(-3)}{5-1} = 5/4$	$(2 - 5/4)/5 = 3/20$	
5	2	$\frac{4-2}{6-5} = 2$		
6	4			

Step 2: write the polynomial

$$P_3(x) = 1 + 2(x-3) - \frac{3}{8}(x-3)(x-1) + \frac{7}{40}(x-3)(x-1)(x-5)$$

Property: If $(z_0, \dots, z_n)$ is a permutation of $(x_0, \dots, x_n)$ then

$$F[x_0, \dots, x_n] = F[z_0, \dots, z_n]$$

both are the coef of x^n in the interp. poly.

HW 6.2: 4, 8, 9, 24

6.3 Hermite Interpolation

Want to interpolate not only the function, but also its derivatives

Setup: at each x_i we are given $P^{(j)}(x_i)$ for $0 \le j \le k_i - 1$

e.g. Find a polynomial P with $P(0) = 0$, $P'(0) = 1$ & $P(1) = 1$

In general: $p^{(j)}(x_i) = c_{ij} \quad (0 \le j \le k_i - 1,\ 0 \le i \le n)$
where $\sum_{i=0}^{n} k_i = m+1$

So, we have $m+1$ conditions $\Rightarrow$ reasonable to look for an mth degree polynomial.

Theorem: $\exists$ a unique polynomial of degree at most m satisfying

$$p^{(j)}(x_i) = c_{ij} \quad (0 \le j \le k_i - 1,\ 0 \le i \le n)$$
where $\sum_{i=0}^{n} k_i = m+1$

Example: Hermite interpolation with only one node.

Given: $p^{(j)}(x_0)$ for $0 \le j \le k$

$$\Rightarrow p(x) = p(x_0) + p'(x_0)(x-x_0) + p''(x_0)(x-x_0)^2/2! + \cdots + p^{(k)}(x_0)\frac{(x-x_0)^k}{k!}$$

Taylor polynomial!

Newton Divided Difference Method

Extensions to some Hermite interpolation

Example: Use extended Newton divided diffs to find a polynomial satisfying

$$P(1)=2,\ P'(1)=3,\ P(2)=6,\ P'(2)=7,\ P''(2)=8$$

Will come back to it.

Method:

x_0, x_0, x_1, x_1, x_1

x_0	$f[x_0]$	$f[x_0,x_0]$	$f[x_0,x_0,x_1]$	$f[x_0,x_0,x_1,x_1]$	$f[-]$
x_0	$f[x_0]$	$f[x_0,x_1]$	$f[x_0,x_1,x_1]$	$f[x_0,x_1,x_1,x_1]$	
x_1	$f[x_1]$	$f[x_1,x_1]$	$f[x_1,x_1,x_1]$		
x_1	$f[x_1]$				
x_1	$f[x_1]$				

$f[x_0,x_0]$

But what is $f[x_0,x_0]$? $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$

$\therefore$ $f[x_0,x_0,x_0]=?$

Apply Theorem (A): $\exists\, \xi = x_0$: $f[x_0,x_0,x_0] = \frac{f''(x_0)}{2}$

Similarly $f[\underbrace{x_0, \dots, x_0}_{k+1 \text{ time}}] = \frac{f^{(k)}(x_0)}{k!}$

OK back to the example

$P(1)=2,\ P'(1)=3,\ P(2)=6,\ P'(2)=7,\ P''(2)=8$

x	$f(x)$				
1	2	3	$\frac{4-3}{1}=1$	$\frac{3-1}{1}=2$	$\frac{1-2}{2-1}=-1$
1	2	$\frac{6-2}{2-1}=4$	$\frac{7-4}{1}=3$	$\frac{4-3}{1}=1$	
2	6	7	$\frac{8}{2!}$		
2	6				
2	6				

Now we use the top row to write $P(x)$

$$P(x) = 2 + 3(x-1) + 1(x-1)^2 + 2(x-1)^2(x-2) - (x-1)^2(x-2)^2$$

Check: $P(1) = 2$ ✓

$P'(1) = 3$ ✓

$P(2) = 2 + 3(2-1) + 1(2-1) + 0 = 6$ ✓

$P'(2) = 3 + 2(2-1) + 2(2-1)^2 = 7$ ✓

$P''(2) = 2 + \dots = 8$ ✓

HW 6.3: 1, 3